Characterization of low-energy mode vibrations in chaos using entropy balance versus the amplitude-based Karhunen-Loéve expansion

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The role played by the low-energy modes in chaos of high-dimensional systems has been studied by using the spatiotemporal entropy balance. The result is a decomposition of the chaotic dynamics into active and passive parts. The active dynamics is responsible for information generation and consumption, whereas the passive part is mostly redundant. We compare our result with the Karhunen-Loéve expansion applied to the same chaotic signal. We found that the mode which appears in the low-energy structure (less than 1% of total energy) can be important to the information dynamics in chaos.

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I. INTRODUCTION

In this paper, we demonstrate two different approaches to study the low-energy mode vibrations in chaos of high-dimensional systems. The first approach is based on the recently proposed *statistical decomposition* (SD) by using the eigenvectors of the *Oseledec matrix* [1,2]. The SD is based on Pesin's (partial and nonuniform) hyperbolicity theory [3–5] and the work by Ledrappier and Young on the extension of Pesin's result on entropy and exponents: the *partial entropy* on the unstable manifolds [6]. These facts result in what we call the *spatiotemporal entropy balance,* which leads to the basis for decomposing chaos by *active* and *passive* degrees of freedom (DOF). The active DOF's are responsible for not only the divergence of orbits on the unstable manifolds but also the convergence on some of the stable manifolds, whereas the *passive* DOF's are redundant in information-theoretic terms. The second approach, which we use for the purpose of comparison, is to apply the Karhunen-Loéve expansion (KL) to the same chaotic signal. The KL expansion is well known in probability theory with many important applications [7–9]. In mechanics, Lumely originally proposed to use KL to describe coherent structures in turbulence flow field [9]. This is achieved by solving eigenfunctions of the two-point covariance kernel of the velocity field. These eigenfunctions will be called the *proper orthogonal modes* (POM). They are arranged in orders based on the corresponding *singular values* that measure the mean-square amplitude (energy) of the projection. The underlying philosophy of KL is amplitude based. When it is applied to derive the finite-dimensional model to approximate the infinite field, it sometimes suffers from the lack of a precise indicator for the relevance of higher-order POM's. The only parameter that can be, and has been, used for deriving such an approximation is the singular value, or the energy, of the POM. In our comparison, we will show that such an exercise may lead to a false impression of the irrelevance of some of the low-energy modes.

To demonstrate our points, we use the model system developed by Cusumano and Moon [10] for the vibration of a thin, elastic beam under harmonic base excitation. Their model has captured two important features observed in the Laboratory: the *three-mode-coupling* scenario originally proposed by Cusumano and Moon and later shown experimentally by Lin (at the instability of planar motion) [11], and the chaotic vibration due to the bending-torsion coupling. Our comparison is focused on the post-torsional-instability chaos of the model system. We found that the low-energy mode that is buried in the POM of less than 1% of total energy can be important in terms of the information dynamics in chaos. Using the SD can successfully pick up such a mode whose importance is precisely indicated.

The organization of this paper is as follows. Brief summaries of the two approaches mentioned above are given in Sec. II. The model system of equations and its finite-dimensional representation are described in Sec. III. Numerical results are shown in Sec. IV, and concluding remarks are given in Sec. V.

II. TWO DIFFERENT APPROACHES

A. Statistical decomposition

The method of SD exploits the space-time relationship resulting from the entropy balance on the (un)stable manifolds. Such a space-time relationship can be expressed by the phase-space coordinates by considering the ensemble of eigenvectors of the Oseledec matrix, which we call the Lyapunov vectors (LV); see [1] and [12–15] for some early studies on Lyapunov vectors.

Consider a $C^{1+\nu}$ vector field f [16] and the dynamical system in $\mathbb{R}^n$:

$$\dot{x} = f(x,t) \tag{1}$$

evolving on a smooth, compact manifold $\mathbf{M}$. Attached to the flow $\phi(x,t)$ generated by (1), there is a tangent map $\mathbf{T}f$, and together they form a *skew-product flow* acting on the tangent bundle $\mathbf{TM} \equiv \cup_x \mathbf{T}_x\mathbf{M}$ [17]. The *characteris-*

tic exponent of $\mathbf{T}f$ along $\phi(x,t)$ can be defined by

$$\lambda(x,v)=\lim_{t\to\infty}\frac{1}{t}\log(\|\mathbf{T}f_x^t v\|) , \tag{2}$$

where $\|\ \|$ denotes Euclidean norm and $\mathbf{T}f_x^t\equiv\mathbf{T}f|_{\phi(x,t)}$. Under the proper conditions [18], one can define *filtrated subspaces* $\{L_x^i\}_{i=1}^n$ on $\mathbf{T}_x\mathbf{M}$ with the corresponding tangent vector growth rate being bounded by the *Lyapunov exponents* $\{\lambda_i\}$: i.e.,

$$L_x^i\equiv\{v\in\mathbf{T}_x\mathbf{M},\ \ \lambda(x,v)\le\lambda_i\} , \tag{3}$$

where λ_i can be obtained from

$$\lambda_i=\lim_{t\to\infty}\frac{1}{t}\log\|\mathbf{T}f_x^t v\|,\quad v\in L_x^i\setminus L_x^{i+1} \tag{4}$$

and $\lambda_1\ge\lambda_2\cdots\ge\lambda_n$. The L_x^i's are invariant, i.e., $\mathbf{T}f_x^t L_x^i=L_{\phi(x,t)}^i$ and

$$\mathbb{R}^n\equiv L_x^1\supseteq L_x^2\supseteq\cdots\supseteq L_x^n\supseteq L_x^{n+1}\equiv\mathbf{0} , \tag{5}$$

where $\mathbf{0}$ stands for the null space. Associated with the $\{L_x^m\}$, one can find a set of (normal) basis vectors $\mathbf{b}\equiv\{V_x^i\}_{i=1}^n$ in $\mathbb{R}^n$ such that $\{V_x^i\}_{i=k}^n$ span the subspace L_x^k and are orthonormal. We call them the *Lyapunov vectors* at x (see [1] for more details).

The relative values of the exponents allow one to read some kind of hyperbolicity of $\mathbf{T}f$ acting on the fibers. In Pesin's theory, the ability to construct $\mathbf{b}$ is crucial, which enables one to establish the notion of *regularity* [4]. For regular orbits, the hyperbolicity of $\mathbf{T}f$ acting on the fibers can further be projected down to $\mathbf{M}$. It was due to Oseledec's multiplicative ergodic theorem, which tells us that, with respect to any invariant Borel probability measure ρ, i.e., $\rho\circ\phi^{-1}=\rho$ [18], almost every point on $\mathbf{M}$ is regular. Pesin then showed what is meant by the hyperbolicity of f: passing through each regular point, there are local (un)stable manifolds [4]. The behavior of points on the (un)stable manifolds manifests properties similar to their counterparts on the tangent spaces. One can then read the dynamics generated by (1) by something similar to (3)–(5) [3,4,6,19]. We should point out that Pesin's measure-theoretic setting applies to the *partially nonuniformly hyperbolic system:* i.e., $\exists v\in\mathbf{T}_x\mathbf{M}$ such that $\lambda(x,v)<0$ [5].

Among other things, Pesin's result suggests a decomposition of the overall complex motion into pieces of random components on the unstable manifolds. Ledrappier and Young were the first to solidify these ideas and develop the notion of *partial entropy* on the unstable foliations [6]. The *entropy balance* of a dynamical system is the direct consequence of these facts [6,20]:

$$\sum_i^n\lambda_i\mathcal{D}_i=\left[\sum^{+}+\sum^{-}+\sum_{i>\alpha}\right]\lambda_i\mathcal{D}_i=0 , \tag{6}$$

where $\mathcal{D}_i$ is the *partial dimension* of the induced conditional measure and the meaning of the index α will be explained shortly. If we denote the ith unstable manifolds by $\mathcal{W}_u^i$, $i>1$, $\mathcal{D}_i$ is the dimension in $\mathcal{W}_u^i\setminus\mathcal{W}_u^{i-1}$; it measures how $\mathcal{W}_u^{i-1}$ is lying in $\mathcal{W}_u^i$ or a "transverse dimension" of $\mathcal{W}_u^{i-1}$ in $\mathcal{W}_u^i$ so that the sum $\sum_k^i\mathcal{D}_k$ yields the dimension of the conditional measure on $\mathcal{W}_u^i$. The same description applies to the $\mathcal{W}_s^{n-i}$ lying in $\mathcal{W}_s^{n-i-1}$; see the Appendix. The sum of the first k terms $\sum_i^k\lambda_i\mathcal{D}_i$ with $\lambda_k>0$ defines the partial entropy on the kth unstable manifolds [6].

The positive part of (6) $\sum^+$ cancels the negative part $\sum^-$, which signifies divergence of the preimages. The amount $\sum^-$ thus represents *information storage rate* on certain stable manifolds. Geometrically, these stable manifolds result in the "asymptotic directions" (of the information dynamics) in the chaotic set [1]. For positive time, the information storage on the "asymptotic" stable manifolds is revealed by the divergence on the unstable manifolds, of exactly $\sum^+$ amount of information generation. In [1], the term *information consumption* was used to describe the dynamics along those stable manifolds associated with the terms in $\sum^-$. For a d-dimensional attractor in $\mathbb{R}^n$ with $n\gg d$, most of the stable directions correspond to the "transient." This fact translates into the $\sum_{i>\alpha}$ terms in the entropy balance equation; i.e., the information generation ($\sum^+$) can be canceled by the consumption terms $\sum^-$ and the rest of the $n-\alpha$ terms are redundant in the information flow in chaos. This is due to $\mathcal{D}_i\approx0$ for $i>\alpha$. Intuitively, it means that there is not enough probability mass (as measured by $\mathcal{D}_i$) to build up the information storage.

If there is a correlation between the phase-space coordinates and the (un)stable manifolds, then the index α that separates the important (un)stable manifolds from the redundant ones can be used to separate the phase-space coordinates into the *active* and *passive* coordinates. As such, chaos can be decomposed into active and passive dynamics with the active dynamics being responsible for generating and consuming information.

The correlation between (un)stable manifolds and the phase-space coordinates can be found from the ensemble of LV's and its distribution [1]. Writing $V_x^i=(v_1,\ldots,v_n)$, the distribution of LV's of a specific exponent is defined by the joint probability density $P_i(v_1,\ldots,v_n)$. In this context, each v_i may either be thought of as a random variable on $[-1,1]$ or the dynamics of the *cocycle* acting on the unit sphere. It is important to note that P_i directly reflects the characteristics of V_i and the rule by which it is applied: $\mathbf{T}f_x^t$. The support of P_i lies *mainly* in the *Lyapunov subspace:* $\mathcal{L}_i\in\mathbb{R}^n$ with the part redundant to the information dynamics being discarded.

We now describe the construction of $\mathcal{L}_i$ (see [1] for more details). It is important to understand *how* the LV's are distributed on the unit sphere in $\mathbb{R}^n$. We use the second-order *cumulant* to achieve this purpose and construct the set of *Lyapunov subspaces* $\{\mathcal{L}_i\}$ whose spanning coordinates, which we call the *correlated coordinates,* characterize the dynamics on the ith unstable manifold [or $(n-i+1)$th stable manifold if $\lambda_i<0$]. Although this characterization is defined by using the local tangent map, it is of global nature since it is derived from the distribution.

The second-order cumulant κ_2 is the covariance and it measures the (linear) correlations of the components of

V_x^i as it "evolves" on the attractor. For an n-dimensional system, the ith Lyapunov vector gives rise to κ_2^i, which is an $n\times n$ covariance matrix. Writing $V_x^i=\sum_{j=1}^{n} v_j \mathbf{e}_j$ where $\|V_x^i\|=1$, the spread of probability mass P_i in $\mathbf{e}_j$ is measured by

$$\kappa_2^i(j,j)\equiv E\{(v_j-\mu_j)^2\}|_{V_x^i} , \quad (7)$$

where $\mu_j\equiv E\{v_j\}$. The (i,j) component of the covariance matrix describes some kind of dependence of the ith and jth coordinates as V_i "evolves" on the unit sphere in $\mathbb{R}^n$. Based on the work by Ledrappier and Young, if $i\le\alpha$, the set of coordinates $\{\mathbf{e}_p\}$ with $\kappa_2^i(p,p)\neq 0$ defines a vector space $\mathcal{L}_i'$ in which the information generation or consumption is taking place. This definition also ensures that any "transient dynamics" such as intermittence be included in the appropriate $\mathcal{L}_i'$. $\mathcal{L}_i$ is derived from $\mathcal{L}_i'$ and it involves the approximation that leaves out the parts in $\mathcal{L}_i'$ that are redundant in the information flow. The case that allows us to do such an approximation is indicated by the condition $\forall i,j$ satisfying $i\le\alpha<j$ and

$$0\approx\kappa_2^i(p,p)<<\kappa_2^j(p,p) . \quad (8)$$

Equation (8) implies that a tangent vector $v\in L_x^1\setminus L_x^{j+1}$ will shrink along the asymptotic direction normal to $\mathbf{e}_p$ since $\lambda_k\ge\lambda_{j+1}$, $k\in[\alpha+1,j]$. On the manifold, because the corresponding $\mathcal{D}_j$ is very small, the information dynamics projected to $\mathbf{e}_p$ is "less observable." Equivalently, the dynamics along V_x^j, which is redundant to the information flow (since $j>\alpha$), is mainly the characteristics of $\mathbf{e}_p$, suggesting it to be regarded as passive. The $\mathbf{e}_p$ will then be dropped from $\mathcal{L}_i'$ for $i\le\alpha$ because of its irrelevance in the information dynamics; i.e., *the information dynamics takes place in a direction transverse to* $\mathbf{e}_p$. In [1], a method has been proposed to check for the transversality of information flows when (8) is indicated. The resulting set of coordinates defines the correlated coordinates and they span the corresponding $\mathcal{L}_i$.

From (6) and the above discussion, we can now form the *active* and *passive* correlated coordinates. The active set is the correlated coordinates of the first α Lyapunov subspaces and the passive set is the correlated coordinates of the remaining Lyapunov subspaces, which have not been included in the active set. In defining these sets, the transversality of certain coordinates of $\mathcal{L}_i'$ values are checked [1]. Finally, α is approximated from $\sum^{\alpha-1}\lambda_i\ge 0$ and $\sum^{\alpha}\lambda_i<0$. This approximation is based on the assumption of a smooth conditional measured on the unstable manifolds [6].

B. Karhunen-Loéve decomposition

The goal of KL theory is to seek the unique expansion of a stochastic process by deterministic functions and statistically uncorrelated coefficients with error minimizing property [7,8,21,22]. Given a second-order stochastic process (bounded covariance) $w(s,t)$, the KL expansion of $w(s,t)$ into the $\mathbf{L}_2$ function space is written as

$$w(s,t)=\sum_n^{\infty}\sqrt{\sigma_n}\,\psi_n(s)z_n(t) , \quad (9)$$

where σ_n is the nth singular value and ψ_n is the corresponding POM. The expansion (9) promises to achieve the following: $(\psi_i,\psi_j)_D=E\{z_iz_j\}=\delta_{ij}$, where $(\ ,\ )$ denotes the inner product of the function space, D, the domain of integration, $E\{\ \}$, the expectation, and δ_{ij} is the Dirac delta function. Hence $w(s,t)$ is decomposed into the sum of product of orthogonal functions ψ_i's and *uncorrelated* random variables z_i's. The σ_n and ψ_n are the eigenvalues and eigenfunctions, respectively, of the correlation kernel:

$$\int_D T_{\text{corr}}(s,s')\psi_n(s')ds'=\sigma_n\psi_n(s) , \quad (10)$$

where $T_{\text{corr}}(s,s')$ is the correlation tensor measured at variable values s and s'. Furthermore, based on the characteristics of T_{corr}, (9) and (10) enjoy the following properties: (a) $\{\sigma_i\}$ is positive and forms a monotonically decreasing sequence, (b) $\{\psi_i\}$ is complete in the space of $\mathbf{L}_2$ functions, and (c) the expansion (9) is unique [7]. When the variables s and t are given physical meanings as space and time, respectively, the decomposition (9) describes the original theme of work of Lumely's (see [22] for a generalization). If s describes the position in some coordinate system (phase space), the first k POM's has the interpretation of the first k principal axes of inertia of some invariant measure carrying $w(s,t)$ in $\mathbb{R}^k$ [21]. The corresponding principal moments of inertia are ordered as $\sigma_1^{-1}<\sigma_2^{-1}\cdots<\sigma_n^{-1}$. In this case, one uses KL for seeking new coordinates $\{\psi_i\}$ by a transformation matrix $\mathcal{A}_{n\times n}$ and ψ_i is the ith column vector of $\mathcal{A}$.

The set $\{\psi_i\}$ is optimal and the error introduced by a k-term expansion of (9) is bounded by σ_{k+1}:

$$\begin{aligned}\|w(s,t)\|&\le\|\sum^{k}\sqrt{\sigma_i}z_i\psi_i\|+\|\sum_{k+1}^{\infty}\sqrt{\sigma_i}z_i\psi_i\|\\&\le\|\sum^{k}\sqrt{\sigma_i}z_i\psi_j\|+\sigma_{k+1} .\end{aligned} \quad (11)$$

The optimality is the minimization of mean-square error of the truncation by using the Lagrange multiplier on the orthogonality of $\{\psi_i\}$ [7,8]: the resulting optimal function set must satisfy (10). Since $\{\sigma_i\}$ form a monotonically decreasing sequence (under given conditions, it can show exponential decay [21]), by choosing k large enough, one is guaranteed of a small enough error. One purpose of this study is to show that this point of view is not sufficient to rationalize the irrelevance of higher-order POM's, $\{\psi_i\}_{i=k+1}^{\infty}$. In particular, it is known that the singular value for intermittent, bursting events in certain coordinate directions can be small in comparison to the others in the complementary subspace. When this happens, such a coordinate is still of crucial importance to model the observed dynamics. It is also possible that certain modes simply oscillate at a much lower amplitude of vibrations but are *directly* related to the information dynamics in chaos. In this latter case, the singular spectrum of such a mode is also small. These cases give the false impression of the irrelevance of the higher-order POM's in the complicated dynamics.

III. APPLICATION TO THE MODEL SYSTEM OF THIN ELASTIC BEAM

We will apply the above ideas to identifying the role played by the low-energy modes in the chaotic vibration of the simplified, nonlinear beam model derived in [10]. Let an overdot denote d/dt and a prime denote $\partial/\partial s$, where s is the spatial variable. The equations of motion read as

$$\ddot{U}+\xi_u\dot{U}+U''''-U\dot{\Phi}^2=\cos(\Phi)F_0\Omega^2\cos(\Omega t)\ , \tag{12a}$$

$$(\mu+U^2)\ddot{\Phi}+\xi_\Phi\dot{\Phi}-\frac{2}{1+\nu}\Phi''+2U\dot{U}\dot{\Phi}=-U\sin(\Phi)F_0\Omega^2\cos(\Omega t)\ , \tag{12b}$$

subjected to the clamped-free boundary conditions

$$U(0,t)=U(0,t)'=U(\bar{l},t)''=U(\bar{l},t)'''=0\ ,$$
$$\Phi(0,t)=\Phi(\bar{l},t)'=0\ , \tag{12c}$$

where U denotes the bending, Φ the torsion, μ,ν the mechanical parameters, $\xi_{u,\Phi}$ the viscous damping coefficients, and F_0 and Ω are the forcing amplitude and frequency, respectively. This system has captured certain important features observed from the experiment, in particular, the nonplanar chaotic vibration [10,11]. This is when the solution $\Phi=\dot{\Phi}\equiv 0$ loses the stability, leading to the subsequent coupled bending-torsion chaos. We derive a seven-mode model of (12), using the first 6 bending and the first torsional modes, from a Galerkin projection to study this behavior [11]. The bases functions are derived from those of the Bernoulli-Euler equation and the torsional wave equation, which are the eigenfunctions of the linearized system based on (12c). The resulting modal equations are given in the following form:

$$\dot{X}=\mathcal{F}(X,t),\quad X\in\mathbb{R}^{14} \tag{13}$$

with the coordinization of X being defined as

$$X=(x_1,\ldots,x_{14})\equiv(u_1,\dot{u}_1,\ldots,u_6,\dot{u}_6,\phi_1,\dot{\phi}_1)\ . \tag{14}$$

In (14), $u_1\cdots u_6$ are the first six bending modes, ϕ_1 is the first torsional mode. Here we make the distinction between the modes and coordinates: the modes are $\{u_1,u_1,\ldots,u_6,\phi_1\}\in\mathbb{R}^7$, and the coordinates are $\{u_1,\dot{u}_1,\ldots,u_6,\dot{u}_6,\phi_1,\dot{\phi}_1\}\in\mathbb{R}^{14}$. The vector field $\mathcal{F}=(f_1,\ldots,f_{14})$ is given by

$$f_{2m-1}=x_{2m}\ ,$$
$$f_{2m}=-\xi_u x_{2m}-\omega_{u_m}^2 x_{2m-1}+\sum_k^6 \mathbf{B}_{mk}x_{2k-1}x_{14}^2+\left[\sum_{k=0,2,4,6}\mathbf{F}_{mk}^u x_{13}^k\right]F_0\Omega^2\cos(\Omega t) \tag{15a}$$

for $m=1,\ldots,6$, and

$$f_{13}=x_{14}\ ,$$
$$f_{14}=\left[-\xi_\phi x_{14}-\mu\omega_{\phi_1}^2 x_{13}-2\sum_i^6\sum_j^6\mathbf{B}_{ij}x_{2i-1}x_{2j}x_{14}-\left[\sum_i^6\sum_{j=1,3,5}\mathbf{F}_{ij}^\phi x_{2i-1}x_{13}^j\right]F_0\Omega^2\cos(\Omega t)\right]\Big/\left[\mu+\sum_i^6\sum_j^6\mathbf{B}_{ij}x_{2i-1}x_{2j-1}\right]\ , \tag{15b}$$

where μ is a constant, ω_{u_i} and ω_{ϕ_1}, the natural frequencies of the bending and the first torsional modes, and $\mathbf{B}$, $\mathbf{F}^u$, and $\mathbf{F}^\phi$ are the coupling and parametric matrices, respectively [1,11]. The modal coupling is controlled by $\mathbf{B}$, $\mathbf{F}^u$, and $\mathbf{F}^\phi$. In particular, $\mathbf{B}$ gives rise to what is known as the *diffusive* coupling; i.e., given i, the entry $\mathbf{B}_{ij}$ decreases monotonically as the value $|i-j|$ increases. Hence all modes are coupled together with the strongest coupling in the nearest neighbors. The harmonic force is applied in the direction orthogonal to the spanwise direction of the beam cross section. As a result, the bending modes are directly excited and the trivial state of the torsion corresponds to an invariant subspace of the system.

While some important, qualitative features of the real beam have been captured by this model (see the Introduction), in what follows, we will concentrate on (13)–(15) as a dynamical system in its own right and apply SD and KL expansion to its chaotic solution.

IV. NUMERICAL RESULTS

We excited the seven-mode model at its third bending resonance, $\Omega=0.537\approx\omega_{u_3}$, and found the complicated chaotic behavior (existence of positive exponent) at $F_0=0.0315$; see Fig. 1. We are particularly interested in the dynamics of small-amplitude modes, i.e., the u_4, u_5, and u_6 modes. By following Wolf *et al.* [23] closely the Lyapunov spectrum is obtained and the largest exponent is ≈ 0.0025 and $\alpha=3$. To identify the set $\{\mathcal{L}_i\}$, assume ergodicity and compute the covariance matrix κ_2^i of the LV. The statistics is based on the sampling from a 15 000 forcing-period simulation. Each κ_2^i is a 14×14 matrix whose (p,q) components measures the *linear* correlation between the p and q coordinates and whose diagonal element $\kappa_2^i(p,p)$ measures the "spread" of P_i in the pth coordinate. In our case, we found that most of the $\kappa_2^i(p,q)$ are small. This may be due to the fact that (7) measures only linear dependence and most of the coordinates are nonlinearly dependent on each other in the information dynamics in the V_x^i direction. The smallness of $\kappa_2^i(p,q)$ can

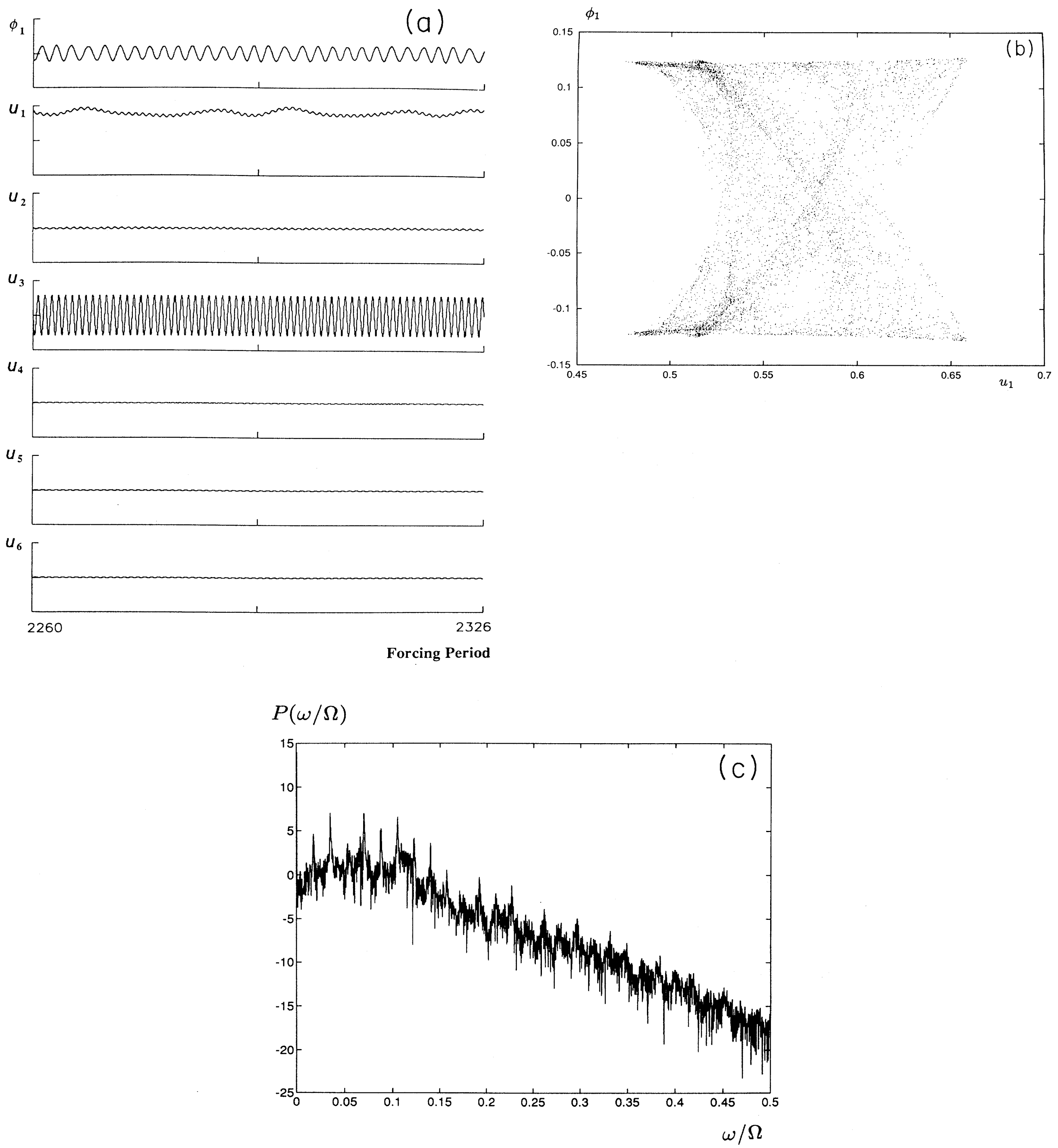

FIG. 1. Nonplanar, chaotic motion at $F_0=0.0315$, $\Omega=0.537$; (a) steady-state modal time series over 70 forcing periods (vertical scale -0.6–0.6), (b) the Poincaré map at the zero phase of the driver projected in (u_1,ϕ_1) plane, (c) the power spectrum of the Poincaré map; all quantities being plotted are dimensionless.

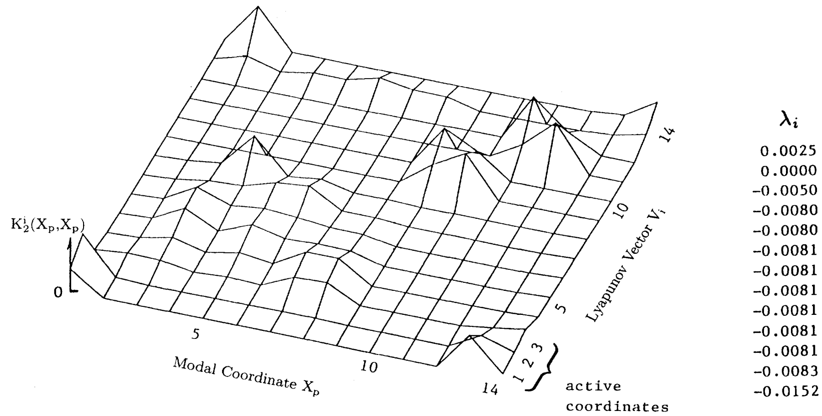

FIG. 2. Diagonal elements of the second-order cumulant of Lyapunov vectors. Note that the zero exponent corresponding to the time axis is not shown here.

also be due to the symmetry in the probability mass of P_i and hence the vanishing of the cross moment of inertia [1]. We thus show only the diagonal elements of the κ_2^i; see Fig. 2. The nonzero $\kappa_2^i(p,p)$ implies the spanning coordinate $\mathbf{e}_p$ of $\mathcal{L}_i'$. For example, Fig. 2 shows that $\mathcal{L}_1'$ is a subspace spanned by x_1, x_{13}, and x_{14}, which we also take as the correlated coordinates of $\mathcal{L}_1$ [1]. This implies the information generation in the V_x^1 direction of the chaotic attractor is due mostly to the dependence of the u_1 and ϕ_1 modes.

Interestingly, u_4 belongs to the first α Lyapunov subspaces; i.e., it is active. Using our terms, its dynamics is related to mostly information consumption. Omission of u_4 in the seven-mode model will lead to qualitatively wrong dynamics (Fig. 3). Furthermore, from $\kappa_2^9 \sim \kappa_2^{12}$, u_5 and u_6 appear to be "disconnected" from the rest. In fact, one can obtain almost the same dynamics without them (see Fig. 3). In [1], we call them the *isolated modes.*

To apply the KL expansion, we first form the correlation tensor T_{corr}. This is given by the 14×14 matrix whose elements are obtained from

$$(T_{\text{corr}})_{ij} = \langle x_i x_j \rangle \ . \tag{16}$$

The International Mathematics and Scientific Library solver DEVCSF has been used to obtain $\{\sigma_i\}$ and $\{\psi_i\}$ of (10) [24] and from which the 14×14 matrix $\mathcal{A}$ is constructed. The result is plotted in Fig. 4 for the components of ψ_i's and the corresponding singular spectrum. It clearly shows that the u_4 (u_5 and u_6) spans the structure (e.g., ψ_7 and ψ_8) of less than 1% of the total energy. As for the modes u_5 and u_6, both methods conclude the same characteristics. We remark that one could also look for the spectrum of *covariance kernel* by *centering* the signals, i.e., the fluctuating parts (see [22] for detailed discussions on this regard). In our test, using the covariance kernel shows more structures in the POM's. But no qualitative difference is seen in terms of the particular POM in which u_4 appears.

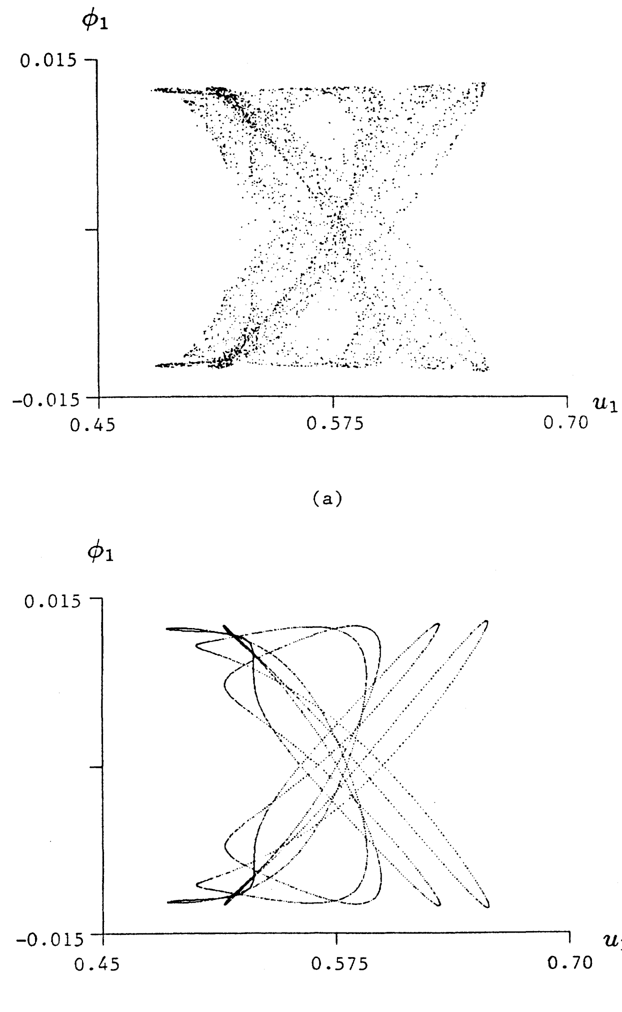

FIG. 3. (a) Projected Poincaré map [on (u_1,ϕ_1) plane] generated by u_1, u_2, u_3, u_4, ϕ_1 modes, (b) projected Poincaré map [on (u_1,ϕ_1) plane] generated by u_1, u_2, u_3, ϕ_1 modes.

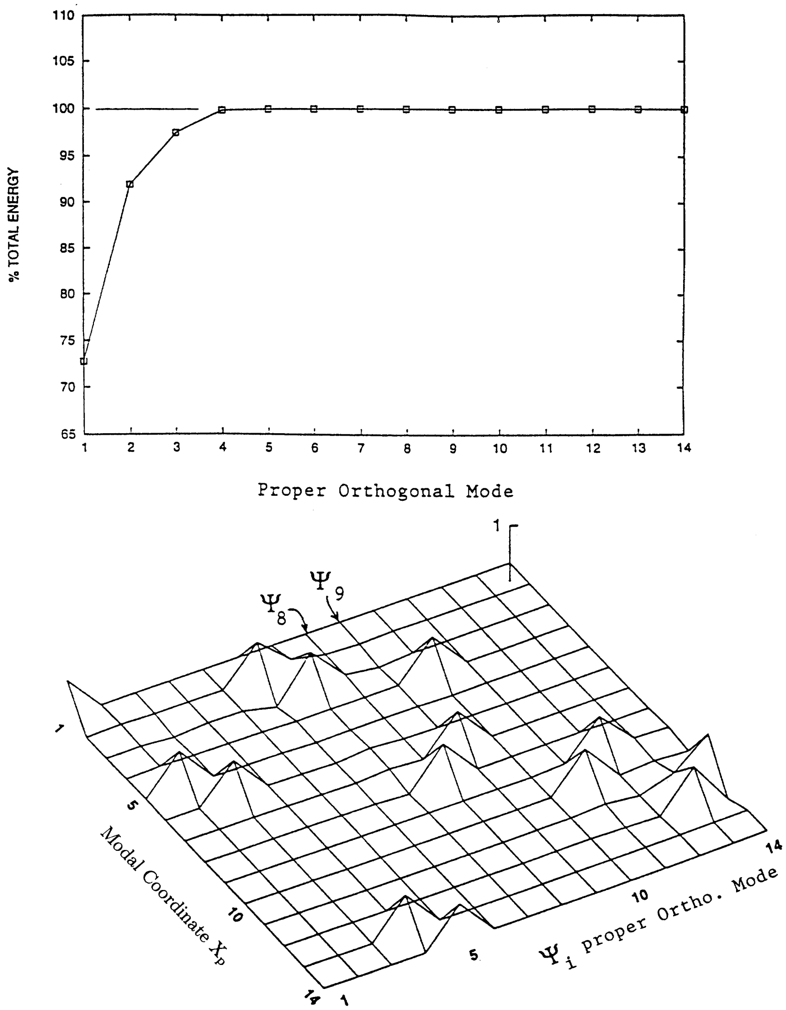

FIG. 4. The 14 POM's generated by applying the Karhunen-Loéve expansion.

V. CONCLUDING REMARKS

Our statistical method shows that *the low-energy modes can be important in terms of the information dynamics in chaos.* The implication is that using the amplitude-based technique *globally,* such as the classical application of Karhunen-Loéve theory, may not be sufficient to model the chaos in high-dimensional systems. For the Karhunen-Loéve method specifically, the possible remedy has been proposed by Broomhead *et al.* [21]. They applied the Karhunen-Loéve expansion only *locally* so as to approximate the tangent spaces. It is quite similar to the ideas of the present approach although we have not made any detailed comparison. The apparent difference is that the current approach is global in nature while the method by Broomhead *et al.* must be completed by a set of connecting matrices to obtain the global result.

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APPENDIX

The discussions of $\mathcal{D}_i$ should start with the important fact resulting from Oseledec's theorem, i.e., the existence of the tangent space decomposition into direct sum of (un)stable "directions" [18]. Under proper conditions, these directions can be "bent down" to produce (un)stable submanifolds of **M**. It is automatic, from (5), that these manifolds are also nested. To be specific, let $m = \max\{i, \lambda_i > 0\}$. The family of unstable manifolds are related by

$$\mathcal{W}_u^m \supseteq \mathcal{W}_u^{m-1} \supseteq \cdots \supseteq \mathcal{W}_u^1 . \tag{A1}$$

The idea of partial dimension is to measure the density of points "in between" the unstable manifolds. To consider such a quantity in the course of finding partial entropy may be motivated by the relationship between metric entropy and dimension [25]. If we make an analogy to statistical physics, the $\mathcal{D}_i$ plays the role of "force variable," which creates the potential for generating entropy [26].

Technically, one needs a *canonical family of measures* and metrics on the unstable manifolds to define $\mathcal{D}_i$. The former can be constructed by standard machinery and the result is unique up to a set of measure zero; see, e.g., Sec. 1.5 in [27]. However, it is necessary to consider measurable partitions *subordinate* to the unstable manifolds. Partitioning the space into unstable manifolds does not quite work because of the complicated fractal structure that makes the partition not separable [6]; see also IV B in [28]. The required metric on the unstable manifold is a result from the Riemannian geometry. If **M** is endowed with a Riemannian metric, there exists an induced metric on $\mathcal{W}_u^i$'s since they are immersed submanifolds; see Sec. 1.2 (example 2.5) in [29]. With all these tools in place, $\mathcal{D}_i$, $i<m$, follows from the standard definition [6]. Detailed proof of its existence and uniqueness is quite involved; Secs. 4 and 11 in [6]. We should leave the details in the reference for a more specific audience interested in this subject.

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